AN ARITHMETIC STUDY OF THE FORMAL LAPLACE TRANSFORM IN SEVERAL VARIABLES.

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ABSTRACT. Let K be a number field, and $\mathcal{F} = K(x_1, \ldots, x_d)$ be the field of rational fractions in the variables $x_1, \ldots, x_d$. In this paper, we introduce two kinds of Laplace transform adapted to solutions of the $\mathcal{F}/K$-differential modules with regular singularities, and give some of their basic differential and arithmetic properties. The purpose of this article is to provide some tools which will be useful in the arithmetic study of $\mathcal{F}/K$-differential modules associated to E-functions in several variables.

1. INTRODUCTION

Let K be a number field, and $\mathcal{F} = K(x_1, \ldots, x_d)$ be the field of rational fractions in the variables $x_1, \ldots, x_d$. Let (M, ∇) be an $\mathcal{F}/K$-differential module of finite rank $\mu \geq 1$, i.e, $M \cong \mathcal{F}^\mu$ and $\nabla : M \to M \otimes_{\mathcal{F}} \Omega^1_{\mathcal{F}/K}$ is an integrable $\mathcal{F}/K$-connection. Assume that there exists a basis $\underline{e}$ of M and matrices $G_1, \ldots, G_d$ of $\mathrm{M}_\mu(K[[x_1, \ldots, x_d]])$ such that

$$\nabla(x_i \frac{\partial}{\partial x_i})\underline{e} = \underline{e} G_i \ , \ \ (i = 1, \ldots, d).$$

Such a differential module is called $\mathcal{F}/K$-differential module with regular singularities.

By recursively iterating of shearing transformations (cf. [4, Proposition 2.3], [5, Lemma 4.3]) with respect to each i, one may assume that the eigenvalues of each $G_i(0)$ do not differ by nonzero integers. According to [9, Corollary p.163], there exists matrix $Y \in \mathrm{GL}_\mu(\overline{K}[[x_1, \ldots, x_d]])$ (where $\overline{K}$ denotes the algebraic closure of K) such that

$$Y^{-1} x_i \frac{\partial}{\partial x_i} Y + Y^{-1} G_i Y = G_i(0), \ \ (i = 1 \ldots, d).$$

The integrability of the connection ∇ ensures that the matrices $G_1(0), \ldots, G_d(0)$ mutually commute, and therefore, the matrix $Y x_1^{G_1(0)} \ldots x_d^{G_d(0)}$ is solution of the system

$$x_i \frac{\partial}{\partial x_i} X = G_i X \ , \ \ (i = 1, \ldots, d). \tag{1.1}$$

In the present paper, we introduce two kinds of Laplace transform $\mathcal{L}$ which extend the two Laplace transformations introduced respectively in [1], [11] and in [10]:

1) the first one (called standard Laplace transform) applies to the entries of $Y x_1^{G_1(0)} \ldots x_d^{G_d(0)}$,

2) the second one (called formal Laplace transform) directly applies to the solution matrix $Y x_1^{G_1(0)} \ldots x_d^{G_d(0)}$.

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These transformations are given under the assumption that all the eigenvalues of $G_1(0),\dots,G_d(0)$ are non-integer numbers. The general case can be reduced to this case by tensorising the $\mathcal{F}/K$-differential module $(\mathcal{M},\nabla)$ with an one-dimensional $\mathcal{F}/K$-differential module associated to the system $\{(x_i\partial/\partial x_i)X=\gamma_i X\}$ for a convenient d-tuple $\underline{\gamma}=(\gamma_1\dots,\gamma_d)$ of $\mathbb{Q}^d$.

These transformations has properties of commutations with the derivations $\partial/\partial x_i$ $(i=1,\dots,d)$ which extend those in one-variable case ((3.12) and (4.13)), and therefore preserve the classic duality between the Laplace transform and the Fourier-Laplace transform. Moreover, for any $\underline{\tau}=(\tau_1,\dots,\tau_d)\in(K\setminus\{0\})^d$, the formal transformation is adapted to have a duality with the generalized Fourier-Laplace transform $\mathcal{F}_{\underline{\tau}}$ with respect to $\underline{\tau}$ (formula (4.13)), defined as the K-automorphism of $K[x_1,\dots,x_d,\partial/\partial x_1,\dots,\partial/\partial x_d]$ determined by:

$$\mathcal{F}_{\underline{\tau}}(x_i)=-\frac{1}{\tau_i}\frac{\partial}{\partial x_i},\quad \mathcal{F}_{\underline{\tau}}(\frac{\partial}{\partial x_i})=\tau_i x_i,\quad (i=1,\dots,d).$$

For $\underline{\tau}=(1,\dots,1)$, $\mathcal{F}_{\underline{\tau}}$ is just the classical Fourier-Laplace transform. In this case, we write $\mathcal{F}$ instead $\mathcal{F}_{\underline{\tau}}$.

The difference between the two transformations is that the standard one involves some transcendental values and applies to terms with logarithms, while the formal one is defined independently of such values and does not apply explicitly to terms with logarithms (4.12).

Under the rationality condition of the eigenvalues of $G_1(0),\dots,G_d(0)$, we prove for each of these transformations its arithmetic properties (Propositions 3.5 and 4.4).

As an application, we show in §3.4 how the standard Laplace transform acts on arithmetic Gevrey series in several variables (Proposition 3.6).

The aim of this paper is to provide some tools which will be useful in the arithmetic study of the $\mathcal{F}/K$-differential module associated to an E-function in several variables using the results of [2], [3] and [7].

2. Notations

Let K be a number field and Σ_f be the set of all finite places v of K. For each $v\in\Sigma_f$ above a prime number $p=p(v)$, we normalize the corresponding v-adic absolute value so that $|p|_v=p^{-1}$ and we put $\pi_v=|p|_v^{1/(p(v)-1)}$. We denote by K_v the v-adic completion of K. We also fix an embedding $K\hookrightarrow\mathbb{C}$.

Let $K((x_1,\dots,x_d))$ be the field of formal laurent series in the variables $x_1,\dots,x_d$, with $d\in\mathbb{N}_{\geq 1}$. Let denote $\underline{x}=(x_1,\dots,x_d)$, $\partial_i=\frac{\partial}{\partial x_i}$, for $i=1,\dots,d$, $\underline{1}=(1,\dots,1)$, let denote by $\underline{\alpha}=(\alpha_1,\dots,\alpha_d)$ the elements of $\mathbb{N}^d$ and $-\underline{\alpha}=(-\alpha_1,\dots,-\alpha_d)$. For $\underline{\alpha},\underline{\beta}\in\mathbb{N}^d$ we put :

$$|\underline{\alpha}|=\sum_{1\leq i\leq d}\alpha_i,\ \ \underline{\alpha}!=\prod_{1\leq i\leq d}\alpha_i!,\ \ \underline{x}^{\underline{\alpha}}=x_1^{\alpha_1}\dots x_d^{\alpha_d},\ \ \underline{\partial}^{\underline{\alpha}}=\partial_1^{\alpha_1}\dots\partial_d^{\alpha_d},$$

$$\underline{\alpha}\leq\underline{\beta}\iff\alpha_i\leq\beta_i\ \text{ for all }\ i=1,\dots,d,\ \text{and}\ \binom{\underline{\alpha}}{\underline{\beta}}=\prod_{1\leq i\leq d}\binom{\alpha_i}{\beta_i}\ \text{ for }\ \underline{\alpha}\geq\underline{\beta}.$$

$$\underline{\alpha}<\underline{\beta}\iff\underline{\alpha}\leq\underline{\beta}\quad\text{and}\ \alpha_i<\beta_i\quad\text{for some}\quad i=1,\dots,d.$$

For a power series $f=\sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}} \in K[[\underline{x}]]$, we denote $f(\frac{1}{\underline{x}})=\sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{-\underline{\alpha}} \in K[[\frac{1}{\underline{x}}]]$. If $v\in\Sigma_f$, we define the radius of convergence $r_v(f)$ of f with respect to v as follows:

$$r_v(f)=\left(\limsup_{|\underline{\alpha}|\to\infty} |a_{\underline{\alpha}}|_v^{1/|\underline{\alpha}|}\right)^{-1}.$$

Finally, for $s\in\mathbb{Z}$, we put

$$\mathcal{R}_v^s(f)=\{y\in K[[\underline{x}]] \mid r_v(y)\geq R_v(f)\ \pi_v^s\}.$$

3. Standard Laplace transform

3.1 Review of the one-variable case. In [1], André extended the classic Laplace-transform $\mathcal{L}$, in the formal way, to logarithmic solutions case in one-variable. Later, in [11], the author gave some arithmetic properties of $\mathcal{L}$. In this paragraph, we recall the definition of $\mathcal{L}(h_{\gamma,k})$ of the term $h_{\gamma,k}:=x^{\gamma}(\log x)^k$ where $(\gamma,k)\in(K\setminus\mathbb{Z}_{\leq 0},\mathbb{N})$, and its arithmetic properties [11, 4] (the case $(\gamma,k)\in(\mathbb{Z}_{\leq 0},\mathbb{N})$ is defined otherwise and has no interest in the definition of $\mathcal{L}$ in several variables as we will see in the next paragraph).

Let us fix an embedding K of into $\mathbb{C}$. Let γ be an element of $K\setminus\mathbb{Z}$ such that $\Re e(\gamma)>-1$, let k and n be two nonnegative integers, and let $h_{\gamma,k}$ denote the function defined by $h_{\gamma,k}(x)=x^{\gamma}(\ln x)^k$; $x>0$. The standard Laplace transform of $h_{\gamma,k}$, denoted $\mathcal{L}(h_{\gamma,k})$, is given by

$$\mathcal{L}(h_{\gamma,k})=\sum_{j=0}^{k}\binom{k}{j}\Gamma^{(j)}(\gamma+1)x^{-\gamma-1}(-1)^{k-j}(\log x)^{k-j}, \tag{3.1}$$

where, $\Gamma^{(j)}$ denotes the j^{th}-derivative of the Euler function Γ (cf. [11, (4.2)]). To extend the Laplace transform $\mathcal{L}$ of $h_{\gamma,k}$ to any $\gamma\in K\setminus\mathbb{Z}$, we have to introduce the functions $F_{\gamma,k,n}$ ([11, (4.8)]), defined for $n\in\mathbb{Z}_{>0}$, by

$$F_{\gamma,k,n}(x)=\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\frac{x^{\gamma+n+1}}{m+\gamma+1}\sum_{\ell=0}^{k}\frac{k!(-1)^{k-\ell}}{\ell!(m+\gamma+1)^{k-\ell}}(\log x)^{\ell}.$$

André observed that the function $x^{n+1}\mathcal{L}(F_{\gamma,k,n})$ is independent of the choice of n for $n\geq-\Re e(\gamma)-1$ (cf. [1, 5.3.6]). According to this remark, the Laplace transform, defined just for $\gamma\in K\setminus\mathbb{Z}$ with $\Re e(\gamma)>-1$ (see (3.1) above), can be extended to any $\gamma\in K\setminus\mathbb{Z}$ by putting

$$\mathcal{L}(h_{\gamma,k}) \quad = \quad z^{n+1}\mathcal{L}(F_{\gamma,k,n}), \quad \text{for} \quad n\geq-\Re e(\gamma)-1. \tag{3.2}$$

Formally, with this definition, we find :

$$\begin{aligned}\mathcal{L}(h_{\gamma,k})=\Gamma(\gamma+1)\ x^{-\gamma-1}\ \sum_{j=0}^{k}\rho_{\gamma,j}^{(k)}\ (\log x)^j\\ \text{with}\ \ \rho_{\gamma,k}^{(k)}=(-1)^k,\ \ \rho_{\gamma,j}^{(k)}\in<\Gamma(\gamma),\ldots,\Gamma^{(k)}(\gamma)>_{\mathbb{Q}[\gamma]},\ \ j=0,\ldots,k-1.\end{aligned} \tag{3.3}$$

Moreover, this transformation $\mathcal{L}$ has the following formal properties (cf. [11, (4.10), (4.11)], [1,

(5.3.7), (5.3.8)]):

$$\frac{d}{dx}(\mathcal{L}(h_{\gamma,k})) = \mathcal{L}(-xh_{\gamma,k}), \quad \text{and} \qquad \mathcal{L}(\frac{d}{dx}h_{\gamma,k}) = x\mathcal{L}(h_{\gamma,k}). \tag{3.4}$$

Notice that the second formula is not valid for $\gamma \in \mathbb{Z}_{\leq 0}$ and $k = 0$ (see formula (4.11) of [11]).

In addition, for each $j = 0, \dots, k$, there exist sequences $\left(r^{(k,\ell)}_{\gamma+n,j}\right)_{n\geq 0}$ of elements of $\mathbb{Q}(\gamma)$, with $\ell = 0, \dots, k$, such that for any $n \in \mathbb{N}$

$$\rho^{(k)}_{\gamma+n,j} = \sum_{\ell=0}^{k} \rho^{(k)}_{\gamma,\ell}\, r^{(k,\ell)}_{\gamma+n,j} \quad (\text{with} \quad r^{(k,\ell)}_{\gamma+n,j} = 0 \quad \text{for} \quad \ell = 0, \dots, j-1). \tag{3.5}$$

Moreover, for any place v of Σ_f, these sequences satisfy

$$\limsup_{n\longrightarrow+\infty} \left| r^{(k,\ell)}_{\gamma+n,j} \right|_v^{1/n} \leq 1. \tag{3.6}$$

According to the proof of Lemma 4.2 [11], one observes that, for all $0 \leq \ell, j \leq k$ and any $n \in \mathbb{N}$, we have:

$$r^{(k,\ell)}_{\gamma+n,j} \in \left\langle 1, \prod_{1\leq i\leq n_t} \frac{1}{\gamma+m_i}, \quad 1 \leq m_i, n_t \leq n, \quad 1 \leq t \leq j \right\rangle_{\mathbb{Z}}. \tag{3.7}$$

Remark 3.1. (i) With the same argument as that employed in the proof of Lemma 4.2 [11], one can prove that the properties above remain valid for $n < 0$, i.e, for each $j = 0, \dots, k$, there exist sequences $\left(r^{(k,\ell)}_{\gamma-n,j}\right)_{n\geq 0}$ of elements of $\mathbb{Q}(\gamma)$, with $\ell = 0, \dots, k$, such that for any $n \in \mathbb{N}$

$$\rho^{(k)}_{\gamma-n,j} = \sum_{\ell=0}^{k} \rho^{(k)}_{\gamma,\ell}\, r^{(k,\ell)}_{\gamma-n,j} \quad (\text{with} \quad r^{(k,\ell)}_{\gamma-n,j} = 0 \quad \text{for} \quad \ell = 0, \dots, j-1). \tag{3.8}$$

and, for any place v of Σ_f, these sequences satisfy

$$\limsup_{n\longrightarrow+\infty} \left| r^{(k,\ell)}_{\gamma-n,j} \right|_v^{1/n} \leq 1. \tag{3.9}$$

Also, for any $0 \leq \ell, j \leq k$ and any $n \in \mathbb{N}$ we have:

$$r^{(k,\ell)}_{\gamma-n,j} \in \left\langle 1, \prod_{1\leq i\leq n_t} \frac{1}{\gamma-m_i}, \quad 0 \leq m_i, n_t \leq n, \quad 1 \leq t \leq j \right\rangle_{\mathbb{Z}}. \tag{3.10}$$

(ii) If $\gamma \in \mathbb{Q} \setminus \mathbb{Z}$, the quantity $r^{(k,\ell)}_{\gamma\pm n,j}$ is a rational number and its denominator has an exponential growth in n.

(iii) The formula (3.3) shows that, for any $(\gamma, k) \in (K \setminus \mathbb{Z}, \mathbb{N})$, $\mathcal{L}(h_{\gamma,k}) \neq 0$, and therefore, by (3.4), we have also $\frac{d}{dx}(\mathcal{L}(h_{\gamma,k})).\mathcal{L}(xh_{\gamma,k}) \neq 0$.

On the other hand, by combining formulae (3.3), (3.5) and (3.8), we find

$$
\begin{aligned}
\mathcal{L}(h_{\gamma+n,k}) &= \Gamma(\gamma+n+1)\; x^{-\gamma-n-1} \sum_{j=0}^{k}\sum_{\ell=0}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma+n,j}^{(k,\ell)}\, (\log x)^j \\
&= (\gamma)_{n+1}\Gamma(\gamma)\; x^{-\gamma-n-1} \sum_{j=0}^{k}\sum_{\ell=0}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma+n,j}^{(k,\ell)}\, (\log x)^j, \\
\mathcal{L}(h_{\gamma-n,k}) &= \Gamma(\gamma-n+1)\; x^{-\gamma+n-1} \sum_{j=0}^{k}\sum_{\ell=0}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma-n,j}^{(k,\ell)}\, (\log x)^j \\
&= \frac{(-1)^n\gamma\Gamma(\gamma)}{(-\gamma)_n}\; x^{-\gamma+n-1} \sum_{j=0}^{k}\sum_{\ell=0}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma-n,j}^{(k,\ell)}\, (\log x)^j,
\end{aligned}
\tag{3.11}
$$

where $(\gamma)_{n+1} = \gamma(\gamma+1)\dots(\gamma+n)$.

3.2 Standard Laplace transform in several variables. Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_d) \in K^d$, $\underline{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and put

$$(\log \underline{x})^{\underline{k}} = (\log x_1)^{k_1} \dots (\log x_d)^{k_d}, \quad h_{\underline{\gamma},\underline{k}} = \underline{x}^{\underline{\gamma}}(\log \underline{x})^{\underline{k}}.$$

In this paragraph, we will try to extend the Laplace transform defined in the previous paragraph to several variables while preserving the commutation with derivations in the following sense:

$$\underline{\partial}^{\underline{\alpha}}(\mathcal{L}(h_{\underline{\gamma},\underline{k}})) = \mathcal{L}((-1)^{|\underline{\alpha}|}\underline{x}^{\underline{\alpha}} h_{\underline{\gamma},\underline{k}}), \quad \text{and} \qquad \mathcal{L}(\underline{\partial}^{\underline{\alpha}} h_{\underline{\gamma},\underline{k}}) = \underline{x}^{\underline{\alpha}}\mathcal{L}(h_{\underline{\gamma},\underline{k}}). \tag{3.12}$$

Notice that these formulae require that for any $1 \le i, j \le d$ with $i \ne j$, $\mathcal{L}(0) = \mathcal{L}(\partial_i(h_{\gamma_j,k_j})) = x_i\mathcal{L}(h_{\gamma_j,k_j})$. This leads to restrict the definition of an eventual $\mathcal{L}$ to terms $h_{\underline{\gamma},\underline{k}}$ such that $(\gamma_i, k_i) \ne (0,0)$ for all $1 \le i \le d$. In addition, for a fixed $(\gamma, k, \alpha) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ with $\alpha \ge max(\gamma, k) + 1$, it arises in the development of $(d/dx)^\alpha(x^\gamma(\log x)^k)$ some monomials with negative exponents. As we have said in the previous paragraph, the second formula in (3.4) (and hence in (3.12)) is not valid for terms $h_{\gamma,0}$ with $\gamma \in \mathbb{Z}_{\le 0}$. For these reasons, we can define the Laplace transform $\mathcal{L}$ only for terms $h_{\underline{\gamma},\underline{k}}$ with $(\underline{\gamma}, \underline{k}) \in (K \setminus \mathbb{Z})^d \times \mathbb{N}^d$.

For $f(\underline{x}) = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{-\underline{\alpha}}\underline{x}^{-\underline{\alpha}} + \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}} \in K[[\underline{x}, \frac{1}{\underline{x}}]]$ and $(\underline{\gamma}, \underline{k}) \in (K \setminus \mathbb{Z})^d \times \mathbb{N}^d$, we define $\mathcal{L}(f(\underline{x})h_{\underline{\gamma},\underline{k}})$ as follows:

$$\mathcal{L}(f(\underline{x})h_{\underline{\gamma},\underline{k}}) = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{-\underline{\alpha}} \prod_{1\le i\le d} \mathcal{L}_i(h_{-\alpha_i+\gamma_i,k_i}) + \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}} \prod_{1\le i\le d} \mathcal{L}_i(h_{\alpha_i+\gamma_i,k_i}),$$

where $\mathcal{L}_i$ denotes the Laplace transform defined in the previous paragraph with respect to the variable x_i. It is easy to check, from (3.4) and (iii) of Remark 3.1, that $\mathcal{L}$ satisfies the formulae of (3.12).

Explicitly, if we put $(\underline{\gamma})_{\underline{\alpha}}=(\gamma_1)_{\alpha_1}\dots(\gamma_d)_{\alpha_d}$ and $\Gamma(\underline{\gamma})=\Gamma(\gamma_1)\dots\Gamma(\gamma_d)$, we get from (3.11):

$$\mathcal{L}(h_{\underline{\gamma}+\underline{\alpha},\underline{k}}) =(\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}}\prod_{1\le i\le d}\Big(\sum_{j_i,\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\ r_{\gamma_i+\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}\Big),$$

$$\mathcal{L}(h_{\underline{\gamma}-\underline{\alpha},\underline{k}}) =\frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})\prod_{1\le i\le d}\gamma_i}{(-\underline{\gamma})_{\underline{\alpha}}}\ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\prod_{1\le i\le d}\Big(\sum_{j_i,\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\ r_{\gamma_i-\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}\Big).$$

Taking into account of (3.3), these equalities can be rewritten as follows:

$$\begin{aligned}\mathcal{L}(h_{\underline{\gamma}+\underline{\alpha},\underline{k}}) &= (-1)^{|\underline{k}|}(\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ h_{-\underline{\gamma}-\underline{\alpha}-\underline{1},\underline{k}}\\ &\quad+\ (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}}\prod_{1\le i\le d}\sum_{\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\sum_{j_i=0}^{k_i-1}r_{\gamma_i+\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}.\\ \mathcal{L}(h_{\underline{\gamma}-\underline{\alpha},\underline{k}}) &= \frac{(-1)^{|\underline{k}|+|\underline{\alpha}|}\Gamma(\underline{\gamma})\prod_{1\le i\le d}\gamma_i}{(-\underline{\gamma})_{\underline{\alpha}}}\ \ h_{-\underline{\gamma}+\underline{\alpha}-\underline{1},\underline{k}}\\ &\quad+\ \frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})\prod_{1\le i\le d}\gamma_i}{(-\underline{\gamma})_{\underline{\alpha}}}\ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\prod_{1\le i\le d}\sum_{\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\sum_{j_i=0}^{k_i-1}r_{\gamma_i-\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}.\end{aligned}\tag{3.13}$$

Remark 3.2. By successive x_i-adic formal completions, the formulae (3.13) show that the Laplace transform $\mathcal{L}$ extends to injective maps:

$$\underline{x}^{\underline{\gamma}}\,K[[\underline{x}]]\hookrightarrow\Gamma(\underline{\gamma})\underline{x}^{-\underline{\gamma}-\underline{1}}\,K[[\frac{1}{\underline{x}}]],\quad \underline{x}^{\underline{\gamma}}\,K[[\frac{1}{\underline{x}}]]\hookrightarrow\Gamma(\underline{\gamma})\underline{x}^{-\underline{\gamma}-\underline{1}}\,K[[\underline{x}]],$$

$$\underline{x}^{\underline{\gamma}}\,K[[\underline{x}]][\log\underline{x}]\hookrightarrow\underline{x}^{-\underline{\gamma}-\underline{1}}\,\mathbb{C}[[\frac{1}{\underline{x}}]][\log\underline{x}],\quad \underline{x}^{\underline{\gamma}}\,K[[\frac{1}{\underline{x}}]][\log\underline{x}]\hookrightarrow\underline{x}^{-\underline{\gamma}-\underline{1}}\,\mathbb{C}[[\underline{x}]][\log\underline{x}].$$

3.3 Arithmetic estimations. Consider the sets

$$\begin{aligned}R(\gamma_i,\alpha_i,k_i)&=\{r_{\gamma_i+\alpha_i,j_i}^{(k_i,\ell_i)}\ |\ 0\le j_i\le k_i,\ 0\le\ell_i\le k_i\ \},\\ R(\gamma_i,-\alpha_i,k_i)&=\{r_{\gamma_i-\alpha_i,j_i}^{(k_i,\ell_i)}\ |\ 0\le j_i\le k_i,\ 0\le\ell_i\le k_i\ \},\\ \rho(\gamma_i,k_i)&=\{\rho_{\gamma_i,\ell_i}^{(k_i)}\ |\ 0\le\ell_i\le k_i\},\end{aligned}$$

each term

$$\prod_{1\le i\le d}\sum_{\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\sum_{j_i=0}^{k_i-1}r_{\gamma_i+\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}\quad\Big(\text{resp.}\prod_{1\le i\le d}\sum_{\ell_i=0}^{k_i}\rho_{\gamma_i,\ell_i}^{(k_i)}\sum_{j_i=0}^{k_i-1}r_{\gamma_i-\alpha_i,j_i}^{(k_i,\ell_i)}\ (\log x_i)^{j_i}\Big)$$

in the right hand side of (3.13) is therefore a finite sum (at most a sum of $\prod_{1\le i\le d}(k_i+1)$ terms) of terms of the form:

$$\begin{aligned}&\Big(\prod_{1\le i\le d}\rho_i\Big)\sum_{|\underline{j}|<|\underline{k}|}\Big(\prod_{1\le i\le d}R_{\alpha_i}^{j_i}\Big)(\log\underline{x})^{\underline{j}}\,,\quad\rho_i\in\rho(\gamma_i,k_i),\quad R_{\alpha_i}^{j_i}\in R(\gamma_i,\alpha_i,k_i)\\ &\Big(\text{resp.}\Big(\prod_{1\le i\le d}\rho_i\Big)\sum_{|\underline{j}|<|\underline{k}|}\Big(\prod_{1\le i\le d}R_{-\alpha_i}^{j_i}\Big)(\log\underline{x})^{\underline{j}}\,,\quad\rho_i\in\rho(\gamma_i,k_i),\quad R_{-\alpha_i}^{j_i}\in R(\gamma_i,-\alpha_i,k_i)\Big).\end{aligned}\tag{3.14}$$

On the other hand, by (3.6), for each fixed i, any sequence $(u_n)_{n\in\mathbb{N}}$, with $u_n\in R(\gamma_i,n,k_i)$ (resp. $u_n\in R(\gamma_i,-n,k_i)$) for all $n\in\mathbb{N}$, satisfies $\limsup_{n\to+\infty}|u_n|_v^{1/n}\le 1$ for any $v\in\Sigma_f$. Moreover, if

$(w_n)_{n\geq 0}$ is a sequence with non-negative integer values such that $w_n \geq n$ for all $n \in \mathbb{N}$, we have $\limsup_{n\to+\infty} |u_n|_v^{1/w_n} \leq 1$ and $\limsup_{n\to+\infty} |u_{n_0}|_v^{1/w_n} \leq 1$ for any fixed index n_0 and any $v \in \Sigma_f$. Combining these remarks, with the properties of elements of $R(\gamma_i, \pm n, k_i)$ (formulae (3.6) and (3.9)), we can say that, for any $v \in \Sigma_f$, we have

$$\tag{3.15} \begin{aligned} &\limsup_{|\underline{\alpha}|\to+\infty} \Big| \prod_{1\leq i\leq d} R_{\alpha_i}^{j_i} \Big|_v^{1/|\underline{\alpha}|} \leq 1, \quad \text{for } R_{\alpha_i}^{j_i} \in R(\gamma_i, \alpha_i, k_i), \quad i = 1, \dots, d \\ &\Big(\text{resp. } \limsup_{|\underline{\alpha}|\to+\infty} \Big| \prod_{1\leq i\leq d} R_{-\alpha_i}^{j_i} \Big|_v^{1/|\underline{\alpha}|} \leq 1, \quad \text{for } R_{-\alpha_i}^{j_i} \in R(\gamma_i, -\alpha_i, k_i), \quad i = 1, \dots, d\Big). \end{aligned}$$

Remark 3.3. According to (ii) Remark 3.1, if $\underline{\gamma} \in (\mathbb{Q}\setminus\mathbb{Z})^d$, the denominator of $\prod_{1\leq i\leq d} R_{\alpha_i}^{j_i}$ (resp. $\prod_{1\leq i\leq d} R_{-\alpha_i}^{j_i}$) has an exponential growth in $|\underline{\alpha}|$.

The arithmetic properties of the new $\mathcal{L}$ are based on the following Lemma which generalizes a well known identity in the p-adic analysis.

Lemma 3.4. *Let $v \in \Sigma_f$. Suppose that $\gamma_1, \dots, \gamma_d$ are (non-liouville) elements of $K \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$. Then*

$$\lim_{|\underline{\alpha}|\to+\infty} \Big| (\underline{\gamma})_{\underline{\alpha}+\underline{1}} \Big|_v^{1/|\underline{\alpha}|} = \pi_v.$$

In particular, for $d = 1$ we have the well known identity $\lim_{\alpha_1\to+\infty} |(\gamma_1)_{\alpha_1}|_v^{1/\alpha_1} = \pi_v$.

Proof. Combining the formulae 7, 12 and 14 of [6], we find that, for any $i = 1, \dots, d$, there exist two real numbers e_i, e_i' such that

$$|p(v)|_v^{(\alpha_i/(p(v)-1)+e_i\log(1+\alpha_i)+e_i')} \leq |\gamma_i^{-1}(\gamma_i)_{(\alpha_i+1)}|_v \leq \frac{|p(v)|_v^{\alpha_i/(p(v)-1)}}{(\alpha_i+1)}, \quad \text{for any } \alpha_i \in \mathbb{N}.$$

Thus,

$$|p(v)|_v^{(|\underline{\alpha}|/(p(v)-1)+\sum_{1\leq i\leq d}(e_i\log(1+\alpha_i)+e_i'))} \Big| \prod_{1\leq i\leq d} \gamma_i \Big|_v \leq |(\underline{\gamma})_{\underline{\alpha}+\underline{1}}|_v \leq \Big| \prod_{1\leq i\leq d} \frac{\gamma_i}{(\alpha_i+1)} \Big|_v |p(v)|_v^{|\underline{\alpha}|/(p(v)-1)},$$

for any $\underline{\alpha} \in \mathbb{N}^d$. Hence, the lemma results from the fact:

$$|\sum_{1\leq i\leq d} e_i \log(1+\alpha_i)| \leq d \max_{1\leq i\leq d}(1, |e_i|) \log(1+|\underline{\alpha}|).$$

$\square$

Proposition 3.5. *Let $v \in \Sigma_f$ and $f = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K[[\underline{x}]]$. Assume $\underline{\gamma} \in (\mathbb{Q}\cap\mathbb{Z}_{p(v)}\setminus\mathbb{Z})^d$. Then there exist power series $f_{\underline{\gamma},\underline{k},\underline{j}} \in \mathbb{C}\otimes_K \mathcal{R}_v^{-1}(f)$ (resp. $f'_{\underline{\gamma},\underline{k},\underline{j}} \in \mathbb{C}\otimes_K \mathcal{R}_v^{1}(f)$), $|\underline{j}| \leq |\underline{k}|$, which satisfy the following conditions*

$$\tag{3.16} \begin{aligned} \mathcal{L}\Big(f h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}} \underline{\Gamma}(\underline{\gamma}) \sum_{|\underline{j}|\leq|\underline{k}|} f_{\underline{\gamma},\underline{k},\underline{j}}\Big(\frac{1}{\underline{x}}\Big)(\log\underline{x})^{\underline{j}} \\ \mathcal{L}\Big(f\Big(\frac{1}{\underline{x}}\Big) h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}} \underline{\Gamma}(\underline{\gamma}) \sum_{|\underline{j}|\leq|\underline{k}|} f'_{\underline{\gamma},\underline{k},\underline{j}}(\log\underline{x})^{\underline{j}} \end{aligned}$$

with $f_{\underline{\gamma},\underline{k},\underline{k}} f'_{\underline{\gamma},\underline{k},\underline{k}} \neq 0$ *such that* $r_v(f_{\underline{\gamma},\underline{k},\underline{k}}) = r_v(f)\pi_v^{-1}$ *(resp.* $r_v(f'_{\underline{\gamma},\underline{k},\underline{k}}) = r_v(f)\pi_v$*). If* $fh_{\underline{\gamma},\underline{k}}$ *(resp.* $f(\frac{1}{\underline{x}})h_{\underline{\gamma},\underline{k}}$*) is solution of a differential equation* ϕ *of* $K[x_1,\ldots,x_d,\partial_1,\ldots,\partial_d]$*, then* $\mathcal{L}(fh_{\underline{\gamma},\underline{k}})$ *(resp.* $\mathcal{L}(f(\frac{1}{\underline{x}})h_{\underline{\gamma},\underline{k}})$*) is solution of* $\mathcal{F}(\phi)$*.*

Proof. The formulae (3.16) follows by combining the formulae (3.13), (3.14), (3.15), and Lemma 3.4. The last statement results from (3.12). □

3.4 Laplace transform and arithmetic Gevrey series. In this paragraph, we shall explain the action of standard Laplace transform in several variables on the arithmetic Gevrey series. Let first give the definition of these power series:

Let $(a_{\underline{\alpha}})$ be a sequence of elements of K. Consider the following conditions:

(A_1): for all $\underline{\alpha} = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$, there exists a constant $C_1 \in \mathbb{R}_{>0}$ such that $a_{\underline{\alpha}}$ and its conjugates over $\mathbb{Q}$ do not exceed $C_1^{|\underline{\alpha}|}$ in absolute value;

(A_2): there exists a constant $C_2 \in \mathbb{R}_{>0}$ such that the common denominator in $\mathbb{N}$ of $\{a_{\underline{\alpha}}, |\underline{\alpha}| \leq n\}$ does not exceed C_2^{n+1}.

Definition. An *arithmetic Gevrey series of order* $s \in \mathbb{Q}$ is an element $f = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}}$ of $K[[\underline{x}]]$ such that the sequence $((\underline{\alpha}!)^{-s}a_{\underline{\alpha}})_{\underline{\alpha}}$ satisfies the conditions (A_1) and (A_2).

The power series $f = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}}$ is called a *G-function* (resp. an *E-function*) if it is an arithmetic Gevrey series of order 0 (resp. of order 1) which satisfies the following holonomy condition:

(H): f is rationally holonomic over $K(\underline{x})$, i.e the $K(\underline{x})/K$-differential module generated by $(D^{\underline{\alpha}}f)_{\underline{\alpha}\in\mathbb{N}^d}$ is a $K(\underline{x})$-vector space of finite dimension.

We conserve the André's notations in [1], and we denote the set of all arithmetic Gevrey series of order $s \in \mathbb{Q}$ in several variables, with coefficients in K, by $K\{\underline{x}\}_s$. Also, we set

$$K\Big\{\frac{1}{\underline{x}}\Big\}_s = \Big\{\sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{-\underline{\alpha}} \mid \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}} \in K\{\underline{x}\}_s\Big\},$$

$$NGA\{\underline{x}\}_s^* = \Big\{\sum_{\text{finite sum}} f_{\underline{\gamma},\underline{k}}h_{\underline{\gamma},\underline{k}} \;\Big|\; f_{\underline{\gamma},\underline{k}} \in K\{\underline{x}\}_s,\ \ (\underline{\gamma},\underline{k}) \in (\mathbb{Q}\setminus\mathbb{Z})^d \times \mathbb{N}^d\Big\},$$

$$NGA\Big\{\frac{1}{\underline{x}}\Big\}_s^* = \Big\{\sum_{\text{finite sum}} f_{\underline{\gamma},\underline{k}}h_{\underline{\gamma},\underline{k}} \;\Big|\; f_{\underline{\gamma},\underline{k}} \in K\Big\{\frac{1}{\underline{x}}\Big\}_s,\ \ (\underline{\gamma},\underline{k}) \in (\mathbb{Q}\setminus\mathbb{Z})^d \times \mathbb{N}^d\Big\}.$$

The elements of $NGA\{\underline{x}\}_s^*$ (resp. $NGA\Big\{\frac{1}{\underline{x}}\Big\}_s^*$) are called Nilson-Gevrey series at 0 (resp. at infinity) of order s. It is clear that these two latter sets are K-vector spaces.

The following result expresses how the Laplace transform $\mathcal{L}$ acts on these K-vector spaces.

Proposition 3.6. *The Laplace transform* $\mathcal{L}$ *induces the following injective* K*-linear maps*

$$NGA\{\underline{x}\}_s^* \hookrightarrow \mathbb{C}\otimes_K NGA\Big\{\frac{1}{\underline{x}}\Big\}_{s+1}^*,$$

$$NGA\Big\{\frac{1}{\underline{x}}\Big\}_{s+1}^* \hookrightarrow \mathbb{C}\otimes_K NGA\{\underline{x}\}_s^*.$$

Proof. This follows by combining formulae (3.13), (3.14), Lemma 3.4 and Remark 3.3. □

4. Formal Laplace transform

4.1. Review of the one-variable case. In this paragraph, we recall the formal Laplace transform in one variable introduced in [10, §5]. This formal transformation allows to avoid the transcendental coefficients as those arising in the Laplace transform seen in the previous section. Moreover, this transformation has properties of commutations with derivation and therefore sends a basis of logarithmic solutions at 0 (resp. at infinity) of a differential equation $\phi \in K[x, d/dx]$ to logarithmic solutions at infinity (resp. at 0) of $\mathcal{F}_\tau(\phi)$ (for any $\tau \in K \setminus \{0\}$).

Let ν be a positive integer, τ an element of $K \setminus \{0\}$ and Λ an $\nu \times \nu$ matrix with entries in K such that all their eigenvalues belongs to $K \setminus \mathbb{Z}$. Then x^Λ is a matrix of $\mathrm{GL}_\nu(K(x^{1/d}, \log x))$ which satisfies

$$\frac{d}{dx}(x^\Lambda) = \Lambda x^{-1} x^\Lambda = \Lambda x^{\Lambda - \mathbb{I}_\nu}.$$

For any integer $\mu \geq 1$ and any $\mu \times \nu$ matrix $Y(x) = \sum_{n=-\infty}^{\infty} Y_i x^n$ with entries in $K[[x, 1/x]]$, we define the *Laplace transform of* $f := Y(x)x^\Lambda$, with respect to Λ and to τ, by

$$\mathcal{L}_\Lambda^\tau(Y(x)x^\Lambda) = \mathcal{L}_\Lambda^\tau\Big(\sum_{n=-\infty}^{\infty} Y_n x^{\Lambda + n\mathbb{I}_\nu}\Big) := \sum_{n=-\infty}^{\infty} Y_n C_{\Lambda,\tau}(n) x^{-\Lambda-(n+1)\mathbb{I}_\nu} \tag{4.1}$$

where $C_{\Lambda,\tau} \colon \mathbb{Z} \to \mathrm{GL}_n(K)$ is defined by the following conditions

$$C_{\Lambda,\tau}(n) = \begin{cases} \tau^{-n}(\Lambda + n\mathbb{I}_\nu)(\Lambda + (n-1)\mathbb{I}_\nu)\cdots(\Lambda + \mathbb{I}_\nu) & \text{si } n \geq 1, \\ \mathbb{I}_\nu & \text{si } n = 0, \\ \tau^{-n}\Lambda^{-1}(\Lambda - \mathbb{I}_\nu)^{-1}\cdots(\Lambda + (n+1)\mathbb{I}_\nu)^{-1} & \text{si } n \leq -1. \end{cases}$$

Notice that $C_{\Lambda,\tau}$, with these properties, satisfies

$$C_{\Lambda,\tau}(n)C_{-\Lambda,\tau}(-n-1) = (-1)^{n+1}\tau\Lambda^{-1}. \tag{4.2}$$

The transformation $\mathcal{L}_\Lambda^\tau$ has the following formal properties [11, 5.1.1]:

$$\mathcal{L}_{-\Lambda}^\tau\big(\mathcal{L}_\Lambda^\tau(f)\big) = -\tau Y(-x)\Lambda^{-1}x^\Lambda, \quad \mathcal{L}_\Lambda^\tau\Big(\frac{df}{dx}\Big) = \tau x \mathcal{L}_\Lambda^\tau(f) \text{ et } \mathcal{L}_\Lambda^\tau(xf) = -\frac{1}{\tau}\frac{d}{dx}\mathcal{L}_\Lambda^\tau(f). \tag{4.3}$$

Moreover, if $\mu = 1$ and if the entries of f are solutions of a differential equation $\phi \in K[x, d/dx]$, then the corresponding of $\mathcal{L}_\Lambda^\tau(f)$ are solutions of $\mathcal{F}_\tau(\phi)$.

4.2 Arithmetic estimations. Let v be a finite place in Σ_f. For any matrix M with entries in K, we denote by $\|M\|_v$ the maximum of the v-adic absolute values of the entries of M. Let $\Lambda \in \mathrm{GL}_\nu(\mathbb{Q})$ be an invertible matrix such that all its eigenvalues lie in $\mathbb{Q} \cap (\mathbb{Z}_{p(v)} \setminus \mathbb{Z})$. In this paragraph, we give upper and lower bounds of $C_{\Lambda,\tau}(n)$ with respect to the norm $\|.\|_v$ for any $n \in \mathbb{Z}$.

Lemma 4.1. *Let $\Lambda \in \mathrm{GL}_\nu(\mathbb{Q})$ with eigenvalues $\gamma_1, \ldots, \gamma_s$ in $\mathbb{Q} \cap (\mathbb{Z}_{p(v)} \setminus \mathbb{Z})$ $(s \leq \nu)$. Then, there exist two positive real numbers c_1, c_2 such that for any $n \geq 1$*

$$c_1 \max_{1\leq j\leq s}\{|\tau^{-n}(\gamma_j + 1)_n|_v\} \leq \|C_{\Lambda,\tau}(n)\|_v \leq c_2 n^{\nu-1} \max_{1\leq j\leq s}\{|\tau^{-n}(\gamma_j + 1)_n|_v\}. \tag{4.4}$$

In particular,

$$\lim_{n\to+\infty} \|C_{\Lambda,\tau}(n)\|_v^{1/n} = |\tau|_v^{-1}\pi_v.$$

Proof. Since the eigenvalues of Λ are all rational numbers, there exists $U \in \mathrm{GL}_n(\mathbb{Q})$ such that the product $\Delta = U^{-1}\Lambda U$ is in Jordan form. Let put $C_{\Delta,\tau} = U^{-1}C_{\Lambda,\tau}U$. Thus, there exist two positive real numbers c_0, c_1, such that $c_1 \leq \|C_{\Lambda,\tau}(n)/C_{\Delta,\tau}(n)\|_v \leq c_0$ for all $n \in \mathbb{Z}$. In addition, the matrix Δ is a block diagonal matrix with blocks $J_1 = \gamma_1\mathbb{I}_{\nu_1} + N_1, \dots, J_s = \gamma_s\mathbb{I}_{\nu_s} + N_s$ on the diagonal (with $\nu_1 + \ldots + \nu_s = \nu$ and $N_1, \dots, N_s$ are nilpotent matrices). Hence, for any $n \in \mathbb{Z}$, $C_{\Delta,\tau}(n)$ is a block diagonal matrix with blocks $C_{J_1,\tau}(n), \dots, C_{J_s,\tau}(n)$ on the diagonal and $\|C_{\Delta,\tau}(n)\|_v = \max_{1\leq j\leq s}\|C_{J_j,\tau}(n)\|_v$. One the other hand, for $j = 1, \dots, s$, we have $(N_j)^\nu = 0$ and for any $n \geq 1$,

$$\begin{aligned} C_{J_j,\tau}(n) &= \tau^{-n}\prod_{\ell=1}^{n}((\gamma_j+\ell)\mathbb{I}_{\nu_j} + N_j) \\ &= \tau^{-n}(\gamma_j+1)_n\Big(\mathbb{I}_\nu + \sum_{t=1}^{\nu-1}\Big(\sum_{1\leq \ell_1<\cdots<\ell_t\leq n}\frac{1}{(\gamma_j+\ell_1)\cdots(\gamma_j+\ell_t)}\Big)N_j^t\Big). \end{aligned} \tag{4.5}$$

Now, for any integer $\ell \geq 1$, the sum $\gamma_j + \ell$ is a rational number for which the absolute value of numerator (in the usual sense) is bounded above by $\theta_j\ell$, for a constant $\theta_j > 0$ which only depends on γ_j, and for which the denominator is prime to p. We deduce $|\gamma_j + \ell|_v \geq (\theta_j\ell)^{-1}$ for any $\ell \geq 1$ and hence, the decomposition (4.5) implies, for any $n \geq 1$,

$$|\tau^{-n}(\gamma_j+1)_n|_v \leq \|C_{J_j,\tau}(n)\|_v \leq (\theta_j n)^{\nu_j-1}|\tau^{-n}(\gamma_j+1)_n|_v.$$

The left inequality results from the fact that all the elements of the diagonal of $C_{J_j,\tau}(n)$ are equals to $\tau^{-1}(\gamma_j+1)_n$. This last observation gives $\det C_{J_j,\tau}(n) = \tau^{-n\nu_j}(\gamma_j+1)_n^{\nu_j}$ and $\det C_{\Lambda,\tau}(n) = \det C_{\Delta,\tau}(n) = \tau^{-n\nu}\prod_{1\leq j\leq s}(\gamma_j+1)_n^{\nu_j}$. Thus,

$$\max_{1\leq j\leq s}\{|\tau^{-n}(\gamma_j+1)_n|_v\} \leq \|C_{\Delta,\tau}(n)\|_v \leq n^{\nu-1}\max_{1\leq j\leq s}\{\theta_j^{\nu_j-1}|\tau^{-n}(\gamma_j+1)_n|_v\},$$

and therefore,

$$c_1\max_{1\leq j\leq s}\{|\tau^{-n}(\gamma_j+1)_n|_v\} \leq \|C_{\Lambda,\tau}(n)\|_v \leq c_0 n^{\nu-1}\max_{1\leq j\leq s}\theta_j^{\nu_j-1}\max_{1\leq j\leq s}\{|\tau^{-n}(\gamma_j+1)_n|_v\}. \tag{4.6}$$

Hence, putting $c_2 = c_0\max_{1\leq j\leq s}\theta_j^{\nu_j-1}$, we get (4.4), and therefore, by Lemma 3.4, the last statement of Lemma 4.1. □

Notice that, for any matrix $Y \in \mathrm{GL}_\nu(K_v)$, we have

$$\|Y\|_v^{-1} \leq \|Y^{-1}\|_v \leq |\det Y|_v^{-1}\|Y\|_v^{\nu-1}. \tag{4.7}$$

Indeed, the relation $\mathbb{I}_\nu = YY^{-1}$ implies $1 = \|\mathbb{I}_\nu\|_v \leq \|Y\|_v\|Y^{-1}\|_v$ which gives the left inequality above. The right inequality comes from the formula $Y^{-1} = (\det Y)^{-1}\mathrm{Adj}(Y)$ where $\mathrm{Adj}(Y)$ denotes the adjoint of Y. Now, by (4.7), if $Z \in \mathrm{GL}_\nu(K_v)$, we have

$$\|Y\|_v|\det Z|_v\|Z\|_v^{1-\nu} \leq \|Y\|_v\|Z^{-1}\|_v^{-1} \leq \|YZ\|_v \leq \|Y\|_v\|Z\|_v. \tag{4.8}$$

Lemma 4.2. *Under the assumptions of Lemma 4.1. There exist two positive real numbers c_3, c_4 such that, for any positive integer $n > 0$, we have*

$$
\begin{aligned}
& c_3(n+1)^{-(1-\nu)^3}|\tau|_v^{-n}(\max_{1\le j\le s}\{|(\gamma_j+1)_{n+1}|_v\})^{-(1-\nu)^2}|\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{\nu-2} \\
& \le \|C_{\Lambda,\tau}(-n)\|_v \le \\
& c_4(n+1)^{(\nu-1)^2}|\tau|_v^{-n}(\max_{1\le j\le s}\{|(\gamma_j+1)_{n+1}|_v\})^{\nu-1}|\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{-1},
\end{aligned}
\tag{4.9}
$$

where $\gamma_1, \dots, \gamma_s$ are the distinct eigenvalues of Λ with multiplicities respectively $\nu_1, \dots, \nu_s$. In particular,

$$
\lim_{n\to+\infty}\|C_{\Lambda,\tau}(-n)\|_v^{1/n} = |\tau|_v\pi_v^{-1}.
$$

Proof. The Lemma (4.1) applies also for $-\Lambda$ instead of Λ since the eigenvalues of $-\Lambda$ belong also to $\mathbb{Q}\cap(\mathbb{Z}_{p(v)}\setminus\mathbb{Z})$. Then, there exist two positive real numbers c_1', c_2', such that, for any integer $n > 0$, we have

$$
c_1'\max_{1\le j\le s}\{|\tau^{-(n+1)}(\gamma_j+1)_{n+1}|_v\} \le \|C_{-\Lambda,\tau}(n+1)\|_v \le c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|\tau^{-(n+1)}(\gamma_j+1)_{n+1}|_v\}.
\tag{4.10}
$$

On the other hand, we have $\det C_{-\Lambda,\tau}(n+1) = \tau^{-(n+1)\nu}\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}$ (see proof of Lemma 4.1), and by (4.2), $C_{\Lambda,\tau}(-n) = \tau\Lambda^{-1}C_{-\Lambda,\tau}(n+1)^{-1}$. Applying (4.8) and (4.7) with $Y = \Lambda^{-1}$ and $Z = C_{-\Lambda,\tau}(n+1)^{-1}$, we get

$$
\begin{aligned}
& \|\Lambda^{-1}\|_v\|C_{-\Lambda,\tau}(n+1)\|_v^{-(1-\nu)^2}|\det C_{-\Lambda,\tau}(n+1)|_v^{\nu-2} \le \|\Lambda^{-1}C_{-\Lambda,\tau}(n+1)^{-1}\|_v \le \\
& \qquad \le \|\Lambda^{-1}\|_v\|C_{-\Lambda,\tau}(n+1)\|_v^{\nu-1}|\det C_{-\Lambda,\tau}(n+1)|_v^{-1}.
\end{aligned}
\tag{4.11}
$$

Replacing now $\det C_{-\Lambda,\tau}(n+1)$ with its value in (4.11) and using (4.10), we get

$$
\begin{aligned}
& |\tau|_v\|\Lambda^{-1}\|_v(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|\tau^{-(n+1)}(\gamma_j+1)_{n+1}|_v\})^{-(1-\nu)^2}|\tau^{-(n+1)\nu}\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{\nu-2} \\
& \le \|C_{\Lambda,\tau}(-n)\|_v \le \\
& |\tau|_v\|\Lambda^{-1}\|_v(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|\tau^{-(n+1)}(\gamma_j+1)_{n+1}|_v\})^{\nu-1}|\tau^{-(n+1)\nu}\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{-1}.
\end{aligned}
$$

or again

$$
\begin{aligned}
& |\tau|_v^n\|\Lambda^{-1}\|_v(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|(\gamma_j+1)_{n+1}|_v\})^{-(1-\nu)^2}|\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{\nu-2} \\
& \le \|C_{\Lambda,\tau}(-n)\|_v \le \\
& |\tau|_v^n\|\Lambda^{-1}\|_v(c_2'(-n-1)^{\nu-1}\max_{1\le j\le s}\{|(\gamma_j+1)_{n+1}|_v\})^{\nu-1}|\prod_{1\le j\le s}(-\gamma_j+1)_{n+1}^{\nu_j}|_v^{-1}.
\end{aligned}
$$

Putting now $c_3 = \|\Lambda^{-1}\|_v(c_2')^{-(1-\nu)^2}$ and $c_4 = \|\Lambda^{-1}\|_v(c_2')^{\nu-1}$, we get (4.9) and, by Lemma 3.4, we obtain the last statement of the Lemma. □

4.3. Formal Laplace transform in several variables. Let $\underline{\tau} = (\tau_1, \dots, \tau_d) \in (K \setminus \{0\})^d$ and let $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_d) \in (\mathrm{GL}_\nu(K))^d$ such that the matrices Λ_i mutually commute and such that all the eigenvalues of $\Lambda_1, \dots, \Lambda_d$ are in $K \setminus \mathbb{Z}$. Put

$$\underline{x}^{\underline{\Lambda}} := x_1^{\Lambda_1} \dots x_d^{\Lambda_d}, \quad \text{and} \quad \underline{\alpha}\mathbb{I}_\nu := (\alpha_1 \mathbb{I}_\nu, \dots, \alpha_d \mathbb{I}_\nu).$$

For any integer $\mu \geq 1$ and any $\mu \times \nu$ matrix $Y(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$ with entries $K[[\underline{x}, 1/\underline{x}]]$, we define the *Laplace transform of* $f := Y(\underline{x})\underline{x}^{\underline{\Lambda}}$ with respect to $\underline{\Lambda}$ and to $\underline{\tau}$ as follows:

(4.12)
$$\begin{aligned}
\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(Y(\underline{x})\underline{x}^{\underline{\Lambda}}) &= \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} \mathcal{L}_{\Lambda_i}^{\tau_i}(x_i^{\Lambda_i + \alpha_i \mathbb{I}_\nu}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) x_i^{-\Lambda_i - (\alpha_i + 1)\mathbb{I}_\nu} \\
&= \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) \underline{x}^{-\underline{\Lambda} - (\underline{\alpha} + \underline{1})\mathbb{I}_\nu} \\
&= \Big(\sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) \underline{x}^{-\underline{\alpha}\mathbb{I}_\nu} \Big) \underline{x}^{-\underline{\Lambda} - \mathbb{I}_\nu} \\
&=: Z_{\underline{\Lambda}}^{\underline{\tau}}(\underline{x}) \underline{x}^{-\underline{\Lambda} - \mathbb{I}_\nu}.
\end{aligned}$$

This definition has a sense, since, by construction, all the matrices $C_{\Lambda_i, \tau_i}(\alpha_i)$ mutually commute for all $\underline{\alpha}$. In addition, according to formula (4.3), we check easily that this Laplace transform commutes with the derivations in the following sense:

$$\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}\Big(\underline{\partial}^{\underline{\beta}}(f)\Big) = \underline{\tau}^{\underline{\beta}} x^{\underline{\beta}} \mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(f) \ \text{ and } \ \mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(\underline{x}^{\underline{\beta}} f) = \frac{(-1)^{|\underline{\beta}|}}{\underline{\tau}^{\underline{\beta}}} \underline{\partial}^{\underline{\beta}}(\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(f)), \quad \text{for any } \underline{\beta} \in \mathbb{N}^d, \tag{4.13}$$

where $\underline{\tau}^{\underline{\beta}} = \tau_1^{\beta_1} \dots \tau_d^{\beta_d}$. This leads to state

Proposition 4.3. *Assume that $\mu = 1$ and that all the entries of f are solutions of a differential equation $\phi \in K[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$. Then, all the entries of $\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(f)$ are solutions of $\mathcal{F}_{\underline{\tau}}(\phi)$.*

According to the notations above, we set

$$Z_{\underline{\Lambda}, \underline{\alpha}}^{\underline{\tau}} = Y_{-\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(-\alpha_i), \quad \text{for all} \quad \underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d,$$

so that

$$Z_{\underline{\Lambda}}^{\underline{\tau}}(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Z_{\underline{\Lambda}, \underline{\alpha}}^{\underline{\tau}} \underline{x}^{-\underline{\alpha}}.$$

The formal transformation $\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}$ has moreover the following arithmetic properties:

Proposition 4.4. *Let v be a finite place in Σ_f. Under the assumptions above, assume that the matrices $\Lambda_1, \dots, \Lambda_d$ belong to $\mathrm{GL}_\nu(\mathbb{Q})$ and that all their eigenvalues are in $\mathbb{Q} \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$. Then*

$$\limsup_{|\underline{\alpha}| \to +\infty} \|Z_{\underline{\Lambda}, \underline{\alpha}}^{\underline{\tau}}\|_v^{1/|\underline{\alpha}|} \leq \pi_v^{-1} |\underline{\tau}|_v \limsup_{|\underline{\alpha}| \to +\infty} \|Y_{-\underline{\alpha}}\|_v^{1/|\underline{\alpha}|},$$

and

$$\limsup_{|\underline{\alpha}| \to +\infty} \|Z_{\underline{\Lambda}, -\underline{\alpha}}^{\underline{\tau}}\|_v^{1/|\underline{\alpha}|} \leq \pi_v |\underline{\tau}|_v^{-1} \limsup_{|\underline{\alpha}| \to +\infty} \|Y_{\underline{\alpha}}\|_v^{1/|\underline{\alpha}|}.$$

Proof. This follows from the fact $\|Z^{\tau}_{\underline{\Lambda},\underline{\alpha}}\|_v \leq \|Y_{-\underline{\alpha}}\|_v \prod_{1\leq i\leq d} \|C_{\Lambda_i,\tau_i}(-\alpha_i)\|_v$ for any $\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d$, and from Lemmas 3.4, 4.1 and 4.2. □

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